

Weighted estimates for multilinear Fourier multipliers*

Kangwei Li and Wenchang Sun[†]

Department of Mathematics and LPMC, Nankai University, Tianjin 300071, China

Email: likangwei9@mail.nankai.edu.cn; sunwch@nankai.edu.cn

Abstract. We prove a Hörmander type multiplier theorem for multilinear Fourier multipliers with multiple weights. We also give weighted estimates for their commutators with vector *BMO* functions.

Keywords. Multiple weights; Multilinear Fourier multipliers

1 Introduction and Main Results

The multilinear Calderón-Zygmund theory was first studied in Coifman and Meyer's works [2, 3]. And it has been widely studied in harmonic analysis by many authors since Lacey and Thiele's work [16, 17] on the bilinear Hilbert transform. For an overview, we refer to [4, 5, 6, 10, 12, 14, 15, 19, 21] and references therein.

In this paper, we study the boundedness of multilinear Fourier multipliers. Specifically, we consider the N -linear Fourier multiplier operator T_m defined by

$$\begin{aligned} T_m(f_1, \dots, f_N)(x) \\ = \int_{\mathbb{R}^{Nn}} e^{2\pi i x \cdot (\xi_1 + \dots + \xi_N)} m(\xi_1, \dots, \xi_N) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N) d\xi_1 \cdots d\xi_N \end{aligned}$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $m \in C^s(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \leq C_{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)} \quad (1.1)$$

for all $|\alpha_1| + \dots + |\alpha_N| \leq s$, and $N \geq 2$ is an integer.

In [2], Coifman and Meyer proved that if s is a sufficient large integer, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$.

In [20], Tomita gave a Hörmander type theorem for multilinear Fourier multipliers. As a consequence, T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$ with $s = \lfloor Nn/2 \rfloor + 1$ in (1.1), where $\lfloor Nn/2 \rfloor$ is the integer part of $Nn/2$. Grafakos and Si [11] gave similar results for the case $p \leq 1$ by using L^r -based Sobolev spaces, $1 < r \leq 2$.

In [8], Fujita and Tomita studied the weighted estimates of T_m under the Hörmander condition and classical A_p weights.

*This work was supported partially by the National Natural Science Foundation of China(10971105 and 10990012).

[†]Corresponding author.

In [18], Lerner, Ombrosi, Pérez, Torres and Trujillo-González introduced the $A_{\vec{P}}$ condition for multiple weights.

Definition 1.1 Let $\vec{P} = (p_1, \dots, p_N)$ with $1 \leq p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. Given $\vec{w} = (w_1, \dots, w_N)$. Set

$$v_{\vec{w}} = \prod_{i=1}^N w_i^{p/p_i}.$$

We say that \vec{w} satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}} \right)^{1/p} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q w_i^{1-p'_i} \right)^{1/p'_i} < \infty. \quad (1.2)$$

When $p_i = 1$, $\left(\frac{1}{|Q|} \int_Q w_i^{1-p'_i} \right)^{1/p'_i}$ is understood as $(\inf_Q w_i)^{-1}$.

In [1], Bui and Duong studied the boundedness of T_m with multiple weights under the condition (1.1). And they also gave a result on commutators.

In this paper, we consider the weighted estimates of T_m with multiple weights. Instead of (1.1), we consider the Hörmander condition. Moreover, we do not assume that s is an integer. To be precise, we prove the following.

Theorem 1.2 Let $\vec{P} = (p_1, \dots, p_N)$ with $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. Suppose that $Nn/2 < s \leq Nn$, that $m \in L^\infty(\mathbb{R}^{Nn})$ with

$$\sup_{R>0} \|m(R\xi)\chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{Nn})} < \infty, \quad (1.3)$$

that $r_0 := Nn/s < p_1, \dots, p_N < \infty$ and that $\vec{w} \in A_{\vec{P}/r_0}$. Then

$$\|T_m(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}. \quad (1.4)$$

Recall that the Sobolev space H^s consists of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H^s} := \|(I - \Delta)^{s/2} f\|_{L^2} < \infty,$$

where $(I - \Delta)^{s/2} f = \mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)^{s/2} \hat{f}(\xi))$. If s is an integer, $\|f\|_{H^s} \asymp \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2}$.

Corollary 1.3 Let $1 < p_1, \dots, p_N < \infty$ with $1/p_1 + \dots + 1/p_N = 1/p$. Suppose that $Nn/2 < s \leq Nn$, that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies (1.3), that $1 < p < \infty$, that $p_\beta := \min\{p_1, \dots, p_N\} < r'_0 = (Nn/s)'$ and that

$$(w_1, \dots, w_{\beta-1}, v_{\vec{w}}^{1-p'}, w_{\beta+1}, \dots, w_N) \in A_{\vec{P}/r_0},$$

where $\vec{P} = (p_1, \dots, p_{\beta-1}, p', p_{\beta+1}, \dots, p_N)$. Then (1.4) holds.

Commutators are a class of non-convolution operators [3, 5, 9, 13]. Here we consider the commutator of a vector BMO function and the multilinear operator with multiple weights. Given a locally integrable vector function $\vec{b} = (b_1, \dots, b_N)$, we define the N -linear commutator of \vec{b} and N -linear operator T_m by

$$T_{m;\vec{b}}(\vec{f}) = \sum_{i=1}^N T_{m;\vec{b}}^i(\vec{f}),$$

where

$$T_{m;\vec{b}}^i(\vec{f}) = b_i T_m(\vec{f}) - T_m(f_1, \dots, b_i f_i, \dots, f_N).$$

If $\vec{b} \in BMO^N$, define $\|\vec{b}\|_{BMO^N} = \sup_{i=1, \dots, N} \|b_i\|_{BMO}$.

Theorem 1.4 *Let the hypotheses be as that in Theorem 1.2. Suppose that $\vec{b} \in BMO^N$. Then*

$$\|T_{m;\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \|\vec{b}\|_{BMO^N} \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}. \quad (1.5)$$

Corollary 1.5 *Let the hypotheses be as that in Corollary 1.3. Suppose that $\vec{b} \in BMO^N$. Then (1.5) holds.*

In the rest of this paper, we give proofs for the above results. We write $A \lesssim B$ if $A \leq CB$ for some positive constant C , depending on N , the dimension n , the Lebesgue exponents and possibly the weights. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.

2 Proof of Main Results

We begin with the definition of Hardy-Littlewood maximal function,

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

The sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $\delta > 0$, we also need the maximal function $M_\delta(f) = M(|f|^\delta)^{1/\delta}$ and $M_\delta^\sharp(f) = M^\sharp(|f|^\delta)^{1/\delta}$.

We use the following form of a classical result by Fefferman and Stein [7].

Proposition 2.1 *Let $0 < p, \delta < \infty$ and $w \in A_\infty$. Then there exists some constant $C_{n,p,\delta,w}$ such that*

$$\int_{\mathbb{R}^n} (M_\delta f)(x)^p w(x) dx \leq C_{n,p,\delta,w} \int_{\mathbb{R}^n} (M_\delta^\sharp f)(x)^p w(x) dx.$$

For $\vec{f} = (f_1, \dots, f_N)$ and $p \geq 1$, we define

$$\mathcal{M}_p(\vec{f}) = \sup_{Q \ni x} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q |f_i(y_i)|^p dy_i \right)^{1/p}.$$

The following proposition gives a necessary and sufficient condition for the boundedness of \mathcal{M}_p .

Proposition 2.2 [1, Proposition 2.3] *Let $p_0 \geq 1$ and $p_i > p_0$ for all $i = 1, \dots, N$ and $1/p_1 + \dots + 1/p_N = 1/p$. Then the inequality*

$$\|\mathcal{M}_{p_0}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}$$

holds if and only if $\vec{w} \in A_{\vec{P}/p_0}$, where $\vec{P}/p_0 = (p_1/p_0, \dots, p_N/p_0)$.

Next we introduce two properties of multiple weights.

Proposition 2.3 [18, Theorem 3.6] *Let $\vec{w} = (w_1, \dots, w_N)$ and $1 \leq p_1, \dots, p_N < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

$$\begin{cases} w_i^{1-p'_i} \in A_{Np'_i}, & i = 1, \dots, N, \\ v_{\vec{w}} \in A_{Np}, \end{cases} \quad (2.1)$$

where the condition $w_i^{1-p'_i} \in A_{Np'_i}$ in the case $p_i = 1$ is understood as $w_i^{1/N} \in A_1$.

The following result appears in [18, Lemma 6.1]. For our purpose, we make a slight change.

Proposition 2.4 *Assume that $\vec{w} = (w_1, \dots, w_N)$ satisfies the $A_{\vec{P}}$ condition, where $\vec{P} = (p_1, \dots, p_N)$ with $1 < p_1, \dots, p_N < \infty$. Let $Nn/2 < s \leq Nn$. Then there exists a constant $1 < r < \min\{p_1, \dots, p_N, s/(s-1), 2s/(Nn)\}$ such that $\vec{w} \in A_{\vec{P}/r}$.*

Proof. Since we need an accurate estimate of r , we sketch the proof of [18, Lemma 6.1]. By using the reverse Hölder inequality, it was shown in [18, Lemma 6.1] that there exist constants $c_i, t_i > 1$ such that

$$\left(\frac{1}{|Q|} \int_Q w_i^{-\frac{t_i}{p_i-1}} \right)^{1/t_i} \leq \frac{c_i}{|Q|} \int_Q w_i^{-\frac{1}{p_i-1}}$$

for all $i = 1, \dots, N$. Let r_i be selected such that

$$\frac{t_i}{p_i - 1} = \frac{1}{\frac{p_i}{r_i} - 1}.$$

Then $r = \min\{r_1, \dots, r_N\}$ satisfies $\vec{w} \in A_{\vec{P}/r}$. By Hölder's inequality, we can choose t_i , and therefore r , be arbitrarily close to 1. Since both $s/(s-1)$ and $2s/(Nn)$ are greater than 1, we get the conclusion desired. \square

The boundedness of multilinear Fourier multipliers was proved in [2, 11, 12, 20]. Here we cite a version in [11].

Proposition 2.5 [11, Theorem 1.1] *Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. Suppose that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies (1.3). Then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

The following lemma is the key to our main results.

Lemma 2.6 *Let the hypotheses be as that in Theorem 1.2 and $0 < \delta < p_0/N$, where $p_0 = rr_0$ and r is the same as that appears in Proposition 2.4. Then for all \vec{f} in product of $L^{q_i}(\mathbb{R}^n)$ spaces with $p_0 \leq q_1, \dots, q_N < \infty$,*

$$M_\delta^\sharp(T_m(\vec{f})) \leq C\mathcal{M}_{p_0}(\vec{f}). \quad (2.2)$$

Proof. By Proposition 2.4, $rr_0 \leq 2$. Consequently, $p_0/N = rr_0/N \leq 1$. Fix a point x and a cube Q such that $x \in Q$. It suffices to prove

$$\left(\frac{1}{|Q|} \int_Q |T_m(\vec{f})(z) - c_Q|^\delta dz \right)^{1/\delta} \leq C\mathcal{M}_{p_0}(\vec{f})(x) \quad (2.3)$$

for some constant c_Q to be determined later since $||\alpha|^\delta - |\beta|^\delta| \leq |\alpha - \beta|^\delta$ for $0 < \delta < 1$. Following the method used in [18], let $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{Q^*}$ for all $i = 1, \dots, N$, and $Q^* = 4\sqrt{n}Q$. Then

$$\begin{aligned} \prod_{i=1}^N f_i(y_i) &= \prod_{i=1}^N (f_i^0(y_i) + f_i^\infty(y_i)) \\ &= \sum_{\alpha_1, \dots, \alpha_N \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \cdots f_N^{\alpha_N}(y_N) \\ &= \prod_{i=1}^N f_i^0(y_i) + \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} f_1^{\alpha_1}(y_1) \cdots f_N^{\alpha_N}(y_N), \end{aligned}$$

where $\mathcal{I} := \{\alpha_1, \dots, \alpha_N : \text{there is at least one } \alpha_i \neq 0\}$. Write then

$$T_m(\vec{f})(z) = T_m(\vec{f}^0)(z) + \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(z) \quad (2.4)$$

Applying Kolmogorov's inequality to the first term

$$T_m(\vec{f}^0)(z) = T_m(f_1^0, \dots, f_N^0)(z),$$

we have

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q |T_m(\vec{f}^0)(z)|^\delta dz \right)^{1/\delta} &\lesssim \|T_m(\vec{f}^0)(z)\|_{L^{p_0/N, \infty}(Q, dx/|Q|)} \\ &\lesssim \prod_{i=1}^N \left(\frac{1}{|Q^*|} \int_{Q^*} |f_i(y_i)|^{p_0} dy_i \right)^{1/p_0} \\ &\leq \mathcal{M}_{p_0}(\vec{f})(x), \end{aligned}$$

since $p_0 > 1$ and T_m is bounded from $L^{p_0} \times \cdots \times L^{p_0}$ to $L^{p_0/N}$, thanks to Proposition 2.5.

In order to study the other terms in (2.4), we set now

$$c = \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(x),$$

and we will show that, for any $z \in Q$, we also get an estimate of the form

$$\sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} |T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(z) - T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(x)| \leq C \mathcal{M}_{p_0}(\vec{f})(x). \quad (2.5)$$

Consider first the case when $\alpha_1 = \cdots = \alpha_N = \infty$ and define

$$T_m(\vec{f}^\infty)(z) = T_m(f_1^\infty, \dots, f_N^\infty)(z).$$

Let $m_j = m(\cdot)\psi(\cdot/2^j)$, where $\psi \in \mathcal{S}(\mathbb{R}^{Nn})$ with $\text{supp } \psi \subset \{\xi \in \mathbb{R}^{Nn} : 1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

We have

$$\begin{aligned} & |T_m(\vec{f}^\infty)(z) - T_m(\vec{f}^\infty)(x)| \\ & \leq \sum_{j \in \mathbb{Z}} |T_{m_j}(\vec{f}^\infty)(z) - T_{m_j}(\vec{f}^\infty)(x)| \\ & \leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{Nn} \setminus (Q^*)^N} |\check{m}_j(z - y_1, \dots, z - y_N) - \check{m}_j(x - y_1, \dots, x - y_N)| \\ & \quad \cdot \prod_{i=1}^N |f_i(y_i)| d\vec{y} \\ & = \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} \int_{(2^{k+1}Q^*)^N \setminus (2^kQ^*)^N} |\check{m}_j(z - y_1, \dots, z - y_N) \\ & \quad - \check{m}_j(x - y_1, \dots, x - y_N)| \cdot \prod_{i=1}^N |f_i(y_i)| d\vec{y} \\ & := \sum_{k=0}^{\infty} I_k. \end{aligned}$$

For any $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} I_k & = \sum_{j \in \mathbb{Z}} \int_{(2^{k+1}Q^*)^N \setminus (2^kQ^*)^N} |\check{m}_j(z - y_1, \dots, z - y_N) \\ & \quad - \check{m}_j(x - y_1, \dots, x - y_N)| \cdot \prod_{i=1}^N |f_i(y_i)| d\vec{y} \\ & \leq \sum_{j \in \mathbb{Z}} \left(\int_{(2^{k+1}Q^*)^N \setminus (2^kQ^*)^N} |\check{m}_j(z - y_1, \dots, z - y_N) \right. \end{aligned}$$

$$\begin{aligned}
& -\check{m}_j(x-y_1, \dots, x-y_N)|^{p'_0} d\vec{y} \Big)^{1/p'_0} \Big(\int_{(2^{k+1}Q^*)^N} \prod_{i=1}^N |f_i(y_i)|^{p_0} d\vec{y} \Big)^{1/p_0} \\
& := \sum_{j \in \mathbb{Z}} J_{j,k} \cdot \Big(\int_{(2^{k+1}Q^*)^N} \prod_{i=1}^N |f_i(y_i)|^{p_0} d\vec{y} \Big)^{1/p_0}.
\end{aligned}$$

Let $h = z - x$ and $\tilde{Q} = x - Q^*$. We have

$$\begin{aligned}
J_{j,k} &= \left(\int_{(2^{k+1}Q^*)^N \setminus (2^kQ^*)^N} |\check{m}_j(z-y_1, \dots, z-y_N) \right. \\
& \quad \left. - \check{m}_j(x-y_1, \dots, x-y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&= \left(\int_{(2^{k+1}\tilde{Q})^N \setminus (2^k\tilde{Q})^N} |\check{m}_j(h+y_1, \dots, h+y_N) - \check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&\leq 2 \left(\int_{c_1 2^k l(Q) \leq |y| \leq c_2 2^k l(Q)} |\check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&\lesssim (2^k l(Q))^{-s} \left(\int_{c_1 2^k l(Q) \leq |y| \leq c_2 2^k l(Q)} (4\pi^2 |y_1|^2 + \dots + 4\pi^2 |y_N|^2)^{sp'_0/2} \right. \\
& \quad \left. \cdot |\check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&\leq (2^k l(Q))^{-s} \left(\int_{\mathbb{R}^{Nn}} (4\pi^2 |y_1|^2 + \dots + 4\pi^2 |y_N|^2)^{sp'_0/2} \right. \\
& \quad \left. \cdot |2^{-jNn} \check{m}_j(2^{-j}y_1, \dots, 2^{-j}y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} 2^{j(Nn/p_0-s)} \\
&\leq (2^k l(Q))^{-s} \left(\int_{\mathbb{R}^{Nn}} (1 + 4\pi^2 |y_1|^2 + \dots + 4\pi^2 |y_N|^2)^{sp'_0/2} \right. \\
& \quad \left. \cdot |2^{-jNn} \check{m}_j(2^{-j}y_1, \dots, 2^{-j}y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} 2^{j(Nn/p_0-s)} \\
&\lesssim (2^k l(Q))^{-s} 2^{j(Nn/p_0-s)} \|m(2^j \cdot) \psi\|_{H^s},
\end{aligned}$$

where the Hausdorff-Young inequality and Hölder inequality are used in the last step since $1 < p_0 \leq 2$ by Proposition 2.4. Suppose that $2^{-l} \leq l(Q) < 2^{-l+1}$. Then we have

$$\begin{aligned}
\sum_{j \geq l} J_{j,k} &\lesssim \sup_j \|m(2^j \cdot) \psi\|_{H^s} \sum_{j \geq l} (2^k l(Q))^{-s} 2^{j(Nn/p_0-s)} \\
&\lesssim \sup_{R>0} \|m(R\xi) \chi_{\{1 < |\xi| < 2\}}\|_{H^s} 2^{-ks} l(Q)^{-Nn/p_0}.
\end{aligned} \tag{2.6}$$

On the other hand, we also have

$$J_{j,k} = \left(\int_{(2^{k+1}\tilde{Q})^N \setminus (2^k\tilde{Q})^N} |\check{m}_j(h+y_1, \dots, h+y_N) - \check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0}$$

$$\begin{aligned}
&\leq \left(\int_{(2^{k+1}\tilde{Q})^N \setminus (2^k\tilde{Q})^N} \left(\int_0^1 |\vec{h} \cdot \nabla \check{m}_j(y_1 + \theta h, \dots, y_N + \theta h)| d\theta \right)^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&\leq \int_0^1 \left(\int_{(2^{k+1}\tilde{Q})^N \setminus (2^k\tilde{Q})^N} |\vec{h} \cdot \nabla \check{m}_j(y_1 + \theta h, \dots, y_N + \theta h)|^{p'_0} d\vec{y} \right)^{1/p'_0} d\theta \\
&\leq \left(\int_{c_1 2^k l(Q) \leq |y| \leq c_2 2^k l(Q)} |\vec{h} \cdot \nabla \check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0},
\end{aligned}$$

where $\vec{h} = (h, \dots, h) \in \mathbb{R}^{Nn}$. Since

$$\vec{h} \cdot \nabla \check{m}_j(y_1, \dots, y_N) = \sum_{r=1}^{Nn} h_r \partial_r \check{m}_j(y_1, \dots, y_N),$$

we have

$$\begin{aligned}
J_{j,k} &\lesssim \sum_{r=1}^{Nn} l(Q) \left(\int_{c_1 2^k l(Q) \leq |y| \leq c_2 2^k l(Q)} |\partial_r \check{m}_j(y_1, \dots, y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \\
&\lesssim \sum_{r=1}^{Nn} l(Q) (2^k l(Q))^{-s} \left(\int_{\mathbb{R}^{Nn}} (1 + 4\pi^2 |y_1|^2 + \dots + 4\pi^2 |y_N|^2)^{sp'_0/2} \right. \\
&\quad \left. \cdot |2^{-jNn} \partial_r \check{m}_j(2^{-j} y_1, \dots, 2^{-j} y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} 2^{j(Nn/p_0 - s)} \\
&\lesssim \sum_{r=1}^{Nn} l(Q) (2^k l(Q))^{-s} 2^{j(Nn/p_0 - s)} 2^j \|m(2^j \xi) \xi_r \psi(\xi)\|_{H^s} \\
&\lesssim \sup_{R>0} \|m(R\xi) \chi_{\{1 < |\xi| < 2\}}\|_{H^s} l(Q) (2^k l(Q))^{-s} 2^{j(Nn/p_0 - s)} 2^j.
\end{aligned}$$

By Proposition 2.4, $Nn/p_0 > s - 1$. It follows that

$$\sum_{j < l} J_{j,k} \lesssim \sup_{R>0} \|m(R\xi) \chi_{\{1 < |\xi| < 2\}}\|_{H^s} 2^{-ks} l(Q)^{-Nn/p_0}. \quad (2.7)$$

Combining the arguments above we get

$$\begin{aligned}
|T_m(f_1^\infty)(z) - T_m(f^\infty)(x)| &\lesssim \sum_{k=0}^{\infty} 2^{-k(s - Nn/p_0)} \mathcal{M}_{p_0}(\vec{f}) \\
&\lesssim \mathcal{M}_{p_0}(\vec{f}).
\end{aligned}$$

What remains to be considered are the terms in (2.5) such that $\alpha_{i_1} = \dots = \alpha_{i_\gamma} = 0$ for some $\{i_1, \dots, i_\gamma\} \subset \{1, \dots, N\}$ and $1 \leq \gamma < N$. We have

$$\begin{aligned}
&|T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(z) - T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(x)| \\
&\leq \sum_j |T_{m_j}(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(z) - T_{m_j}(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(x)| \\
&\leq \sum_j \prod_{i \in \{i_1, \dots, i_\gamma\}} \int_{Q^*} |f_i(y_i)| dy_i \int_{(\mathbb{R}^n \setminus Q^*)^{N-\gamma}} |\check{m}_j(z - y_1, \dots, z - y_N)|
\end{aligned}$$

$$\begin{aligned}
& -\check{m}_j(x - y_1, \dots, x - y_N) \prod_{i \notin \{i_1, \dots, i_\gamma\}} |f_i(y_i)| dy_i \\
&= \sum_j \sum_{k=0}^{\infty} \prod_{i \in \{i_1, \dots, i_\gamma\}} \int_{Q^*} |f_i(y_i)| dy_i \int_{(2^{k+1}Q^* \setminus 2^k Q^*)^{N-\gamma}} |\check{m}_j(z - y_1, \dots, z - y_N) \\
&\quad -\check{m}_j(x - y_1, \dots, x - y_N)| \prod_{i \notin \{i_1, \dots, i_\gamma\}} |f_i(y_i)| dy_i \\
&\leq \sum_j \sum_{k=0}^{\infty} \left(\int_{(Q^*)^\gamma \times (2^{k+1}Q^* \setminus 2^k Q^*)^{N-\gamma}} |\check{m}_j(z - y_1, \dots, z - y_N) \right. \\
&\quad \left. -\check{m}_j(x - y_1, \dots, x - y_N)|^{p'_0} d\vec{y} \right)^{1/p'_0} \left(\int_{(2^{k+1}Q^*)^N} \prod_{i=1}^N |f_i(y_i)|^{p_0} d\vec{y} \right)^{1/p_0}.
\end{aligned}$$

Then by similar arguments as above we get that

$$|T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(z) - T_m(f_1^{\alpha_1}, \dots, f_N^{\alpha_N})(x)| \leq C \mathcal{M}_{p_0}(\vec{f})(x).$$

This completes the proof. \square

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 2.1 and Lemma 2.6, we have

$$\begin{aligned}
\|T_m(\vec{f})\|_{L^p(v_{\vec{w}})} &\leq \|M_\delta(T_m(\vec{f}))\|_{L^p(v_{\vec{w}})} \leq C_{n,p,\delta,\vec{w}} \|M_\delta^\sharp(T_m(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\leq C \|\mathcal{M}_{p_0}(\vec{f})\|_{L^p(v_{\vec{w}})}.
\end{aligned}$$

Now the desired conclusion follows from Proposition 2.2. \square

Proof of Corollary 1.3. Assume that $p_1 = \min\{p_1, \dots, p_N\}$ without loss of generality. As in [8], we set

$$m_1 = m(-(\xi_1 + \dots + \xi_N), \xi_2, \dots, \xi_N).$$

Then we can write

$$\int_{\mathbb{R}^n} T_m(f_1, \dots, f_N) g dx = \int_{\mathbb{R}^n} T_{m_1}(g, f_2, \dots, f_N) f_1 dx \quad (2.8)$$

for all $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. By a change of variables we get

$$\sup_{R>0} \|m_1(R\xi) \chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{Nn})} \leq C \sup_{R>0} \|m(R\xi) \chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{Nn})} < \infty.$$

Since $1/p_1 + \dots + 1/p_N = 1/p$, we have $1/p' + 1/p_2 + \dots + 1/p_N = 1/p'_1$. Therefore, $Nn/s < \min\{p', p_2, \dots, p_N\}$ due to $\min\{p_1, \dots, p_N\} < (Nn/s)'$. Since

$$v_{\vec{w}}^{(1-p')p'_1/p'} w_2^{p'_1/p_2} \dots w_N^{p'_1/p_N} = w_1^{1-p'_1},$$

it follows from Theorem 1.2 that

$$\|T_{m_1}(\vec{f})\|_{L^{p'_1}(w_1^{1-p'_1})} \leq C \|f_1\|_{L^{p'}(v_{\vec{w}}^{1-p'})} \prod_{i=2}^N \|f_i\|_{L^{p_i}(w_i)}. \quad (2.9)$$

Then by duality and (2.8) and (2.9), we have

$$\begin{aligned}
\|T_m(\vec{f})\|_{L^p(v_{\vec{w}})} &= \sup_{\|g\|_{L^{p'}(v_{\vec{w}}^{1-p'})}=1} |\langle T_m(f_1, \dots, f_N), g \rangle| \\
&= \sup_{\|g\|_{L^{p'}(v_{\vec{w}}^{1-p'})}=1} |\langle T_{m_1}(g, \dots, f_N), f_1 \rangle| \\
&\leq \sup_{\|g\|_{L^{p'}(v_{\vec{w}}^{1-p'})}=1} \|T_{m_1}(g, f_2, \dots, f_N)\|_{L^{p'_1}(w_1^{1-p'_1})} \|f_1\|_{L^{p_1}(w_1)} \\
&\leq C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}.
\end{aligned}$$

This completes the proof. \square

Lemma 2.7 *Suppose that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies (1.3), that $\vec{b} \in BMO^N$ and that $0 < \delta < \epsilon < p_0/N$. Then for any $q_0 > p_0$, there exists some constant $C > 0$ such that*

$$M_\delta^\sharp(T_{m;\vec{b}}(\vec{f}))(x) \leq C \|\vec{b}\|_{BMO^N} (M_\epsilon(T_m(\vec{f}))(x) + \mathcal{M}_{q_0}(\vec{f})(x))$$

for all N -tuples $\vec{f} = (f_1, \dots, f_N)$ of bounded measurable functions with compact support.

Proof. By linearity it suffices to consider the case of $\vec{b} = b \in BMO$. Fix $b \in BMO$ and consider the operator

$$T_{m;b}(\vec{f}) = bT_m(\vec{f}) - T_m(bf_1, f_2, \dots, f_N).$$

Fix $x \in \mathbb{R}^n$. For any cube Q centered at x , set $Q^* = 4\sqrt{n}Q$. Then we have

$$T_{m;b}(\vec{f}) = (b - b_{Q^*})T_m(\vec{f}) - T_m((b - b_{Q^*})f_1, f_2, \dots, f_N).$$

Since $0 < \delta < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|} \int_Q \left| |T_{m;b}(\vec{f})(z)|^\delta - |c|^\delta \right| dz \right)^{1/\delta} \\
&\leq \left(\frac{1}{|Q|} \int_Q |T_{m;b}(\vec{f})(z) - c|^\delta dz \right)^{1/\delta} \\
&\lesssim \left(\frac{1}{|Q|} \int_Q |(b - b_{Q^*})T_m(\vec{f})(z)|^\delta dz \right)^{1/\delta} \\
&\quad + \left(\frac{1}{|Q|} \int_Q |T_m((b - b_{Q^*})f_1, f_2, \dots, f_N)(z) - c|^\delta dz \right)^{1/\delta} \\
&:= I + II.
\end{aligned}$$

For any $1 < q < \epsilon/\delta$, by Hölder and John-Nirenberg inequality, we have

$$I \leq \left(\frac{1}{|Q|} \int_Q |b - b_{Q^*}|^{q'\delta} dz \right)^{1/q'\delta} \left(\frac{1}{|Q|} \int_Q |T_m(\vec{f})(z)|^{q\delta} dz \right)^{1/q\delta}$$

$$\lesssim \|b\|_{BMO} M_\epsilon(T_m(\vec{f}))(x)$$

Using the similar decomposition as that in the proof of Lemma 2.6, we can write

$$\begin{aligned} \prod_{i=1}^N f_i(y_i) &= \sum_{\alpha_1, \dots, \alpha_N \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \cdots f_N^{\alpha_N}(y_N) \\ &= \prod_{i=1}^N f_i^0(y_i) + \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} f_1^{\alpha_1}(y_1) \cdots f_N^{\alpha_N}(y_N), \end{aligned}$$

Let $c = \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} T_m((b - b_{Q^*})f_1, f_2, \dots, f_N)(x)$. We have

$$\begin{aligned} II &\lesssim \left(\frac{1}{|Q|} \int_Q |T_m((b - b_{Q^*})f_1^0, f_2^0, \dots, f_N^0)(z)|^\delta dz \right)^{1/\delta} \\ &\quad + \sum_{\alpha_1, \dots, \alpha_N} \left(\frac{1}{|Q|} \int_Q |T_m((b - b_{Q^*})f_1^{\alpha_1}, f_2^{\alpha_2}, \dots, f_N^{\alpha_N})(z) \right. \\ &\quad \left. - T_m((b - b_{Q^*})f_1^{\alpha_1}, f_2^{\alpha_2}, \dots, f_N^{\alpha_N})(x)|^\delta dz \right)^{1/\delta} \\ &= II_1 + \sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} II_{\alpha_1, \dots, \alpha_N}. \end{aligned}$$

We first estimate II_1 . By Kolmogorov's and Hölder's inequalities, we have

$$\begin{aligned} II_1 &\lesssim \|T_m((b - b_{Q^*})f_1^0, f_2^0, \dots, f_N^0)\|_{L^{p_0/N, \infty}(Q, dx/|Q|)} \\ &\lesssim \left(\frac{1}{|Q^*|} \int_{Q^*} |(b - b_{Q^*})f_1(z)|^{p_0} dz \right)^{1/p_0} \prod_{i=2}^N \left(\frac{1}{|Q^*|} \int_{Q^*} |f_i(z)|^{p_0} dz \right)^{1/p_0} \\ &\lesssim \|b\|_{BMO} \mathcal{M}_{q_0}(\vec{f})(x). \end{aligned}$$

Next we estimate $II_{\alpha_1, \dots, \alpha_N}$. By similar arguments as that in the proof of Lemma 2.6, we have

$$\begin{aligned} &\sum_{\alpha_1, \dots, \alpha_N \in \mathcal{I}} II_{\alpha_1, \dots, \alpha_N} \\ &\lesssim \sum_{k=0}^{\infty} 2^{-k(s - Nn/p_0)} \left(\frac{1}{|2^{k+1}Q^*|} \int_{2^{k+1}Q^*} |(b - b_{Q^*})f_1(y_1)|^{p_0} dy_1 \right)^{1/p_0} \\ &\quad \cdot \prod_{i=2}^N \left(\frac{1}{|2^{k+1}Q^*|} \int_{2^{k+1}Q^*} |f_i(y_i)|^{p_0} dy_i \right)^{1/p_0} \\ &\lesssim \|b\|_{BMO} \mathcal{M}_{q_0}(\vec{f})(x). \end{aligned}$$

This completes the proof. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 2.4, there is some $1 < r' < \min\{p_1/p_0, \dots, p_N/p_0\}$ such that $\vec{w} \in A_{\vec{P}/(p_0 r')}$. Let $q_0 = p_0 r'$, by Proposition 2.2,

$$\|\mathcal{M}_{q_0}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}.$$

By Proposition 2.1 and Lemma 2.6,

$$\begin{aligned} \|M_\epsilon(T_m(\vec{f}))\|_{L^p(v_{\vec{w}})} &\leq C_{n,p,\delta,\vec{w}} \|M_\epsilon^\sharp(T_m(\vec{f}))\|_{L^p(v_{\vec{w}})} \\ &\leq C \|\mathcal{M}_{p_0}(\vec{f})\|_{L^p(v_{\vec{w}})} \end{aligned}$$

Then the desired conclusion follows from Proposition 2.2 and Lemma 2.7. \square

Proof of Corollary 1.5. By linearity it is enough to consider the case of $\vec{b} = b \in BMO$. Fix $b \in BMO$ and consider the operator

$$T_{m;b}(\vec{f}) = bT_m(\vec{f}) - T_m(bf_1, f_2, \dots, f_N).$$

Notice that

$$\begin{aligned} &\int_{\mathbb{R}^n} T_{m;b}(f_1, \dots, f_N) g dx \\ &= \int_{\mathbb{R}^n} T_{m_1}(bg, f_2, \dots, f_N) f_1 dx - \int_{\mathbb{R}^n} T_{m_1}(g, f_2, \dots, f_N) b f_1 dx \\ &= - \int_{\mathbb{R}^n} T_{m_1;b}(g, \dots, f_N) f_1 dx. \end{aligned}$$

By similar arguments as that in the proof of Corollary 1.3 we can get the conclusion desired. \square

References

- [1] T.A. Bui, X.T. Duong, Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers, <http://arxiv.org/abs/1112.0823v2>
- [2] R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, *Astérisque* **57**(1978), 1–185.
- [3] R. Coifman and Y. Meyer, On commutators of singular integral and bilinear singular integrals, *Trans. Amer. Math. Soc.* **212**(1975), 315–331.
- [4] Y. Ding, S. Lu, D. Yang, A criterion on weighted L^p boundedness for rough multilinear oscillatory singular integrals, *Proc. Amer. Math. Soc.*, **129** (2001), 1127–1136.
- [5] X.T. Duong, L. Grafakos and L. Yan, Multilinear operators with non-smooth kernels and commutators of singular integrals, *Trans. Amer. Math. Soc.* **362**(2010), 2089–2113.
- [6] X.T. Duong, R. Gong, L. Grafakos, J. Li and L. Yan, Maximal operator for multilinear singular integrals with non-smooth kernels, *Indiana Univ. Math. J.* **58**(2009), 2517–2541.

- [7] C. Fefferman, E.M. Stein, H^p spaces of several variables, *Acta Math.* **129**(1972), 137–193.
- [8] M. Fujita and N. Tomita, Weighted norm inequalities for multilinear Fourier multipliers, *Trans. Amer. Math. Soc.* (2012) In Press.
- [9] L. Grafakos, Classical Fourier Analysis(second ed.), Springer-Verlag, 2008.
- [10] L. Grafakos, L. Liu and D. Yang, Multiple weighted norm inequalities for maximal multilinear singular integrals with non-smooth kernels, *Proceedings of the Royal Society of Edinburgh* **141A**(2011), 755–775.
- [11] L. Grafakos and Z. Si, The Hörmander type multiplier theorem for multilinear operators, *Journal fur die Reine und Angewandte Mathematik*, In press.
- [12] L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, *Adv. Math.* **165**(2002), 124–164.
- [13] G. Hu, Y. Meng, D. Yang, Multilinear commutators of singular integrals with non doubling measures, *Integral Equations Operator Theory*, **51** (2005), 235–255.
- [14] G. Hu and D. Yang, A variant sharp estimate for multilinear singular integral operators, *Studia Math.* **141**(2000), 25–42.
- [15] G. Hu and D. Yang, Sharp function estimates and weighted norm inequalities for multilinear singular integral operators, *Bull. London Math. Soc.*, **35** (2003), 759–769.
- [16] M. Lacey, C. Thiele, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. of Math.*, **146** (1997), 693–724.
- [17] M. Lacey, C. Thiele, On Calderóns conjecture, *Ann. of Math.*, **149** (1999), 475–496.
- [18] A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. in Math.*, **220**(2009), 1222–1264.
- [19] B. Ma and M.W. Wong, L^p -boundedness of wavelet multipliers, *Hokkaido Math. J.* **33**(2004), 637–645.
- [20] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, *J. Funct. Anal.* **259**(2010), 2028–2044.
- [21] D. Yang, W. Yuan, and C. Zhuo, Fourier multipliers on Triebel-Lizorkin-type spaces, *J. Funct. Spaces and Appl.* In Press. doi:10.1155/2012/431016